THE UNIQUENESS OF HERMITE SERIES UNDER POISSON-ABEL SUMMABILITY(1)

BY
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Introduction. The differential equation

$$y''(x)-2xy'(x)+2nyx = 0 \qquad (n = 0, 1, 2, \ldots), \tag{1}$$

known as Hermite's equation arises in quantum mechanics in connection with the one-dimensional simple harmonic oscillator. It can be solved by the method of undetermined coefficients and yields the Hermite polynomials

$$H_n(x) = (-1)^n \exp(x^2)[d^n \exp(-x^2)/dx^n] \qquad (n = 0, 1, 2, \ldots). \tag{2}$$

These polynomials are orthogonal with respect to the weight function $\exp(-x^2)$. We will work with the normalized Hermite functions

$$\Phi_n(x) = \exp(-x^2/2)H_n(x)/\pi^{1/4}(n!)^{1/2}2^{n/2} \qquad (n = 0, 1, 2, \ldots), \tag{3}$$

which are orthonormal on $(-\infty, \infty)$ and complete in $\mathscr{L}_2$ [1, p. 288].

Notations and definitions will be as in the main reference of this paper [2]. For any real number p, we say that $f \in \mathscr{H}_p$ if the function $f \in \mathscr{L}$ on every finite interval, and if

$$\int_{-\infty}^{\infty} |x^p f(x)| \exp(-x^2/2)\, dx < +\infty. \tag{4}$$

If $f \in \mathscr{H}_p$, for every $p \geqq 0$, we say that $f \in \mathscr{H}$. If $f \in \mathscr{H}$, and if

$$a_n = \int_{-\infty}^{\infty} f(x)\Phi_n(x)\, dx \qquad (n = 0, 1, 2, \ldots),$$

the generalized Fourier series coefficients, then we say that the series $\sum_{n=0}^{\infty} a_n\Phi_n(x)$ is the Hermite series of $f(x)$, and write

$$f(x) \sim \sum_{n=0}^{\infty} a_n\Phi_n(x). \tag{5}$$

If the series $\sum_{n=0}^{\infty} a_n\Phi_n(x)r^n$ converges, for $0 \leqq r < 1$, to $f(x, r)$, then the functions $f^*(x) = \lim\sup_{r\to 1} f(x, r)$ and $f_*(x) = \lim\inf_{r \to 1} f(x, r)$ are called the upper and

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lower Poisson sums respectively of the series $\sum_{n=0}^{\infty} a_n\Phi_n(x)$. Analogous to the Riemann function for trigonometric series [3, p. 319], we form the series

(6) $$F(x, r) = -\sum_{n=0}^{\infty} \frac{a_n}{2n+2}\Phi_n(x)r^n \qquad (0 \leqq r < 1)$$

which under the given hypothesis converges to a function $F(x)$.

Major theorems. This paper is concerned with the problem of determining when a given series of Hermite functions, $\sum_{n=0}^{\infty} a_n\Phi_n(x)$, is a Hermite series in the above sense. In Chapter II we prove

THEOREM I. *Let the series* $\sum_{n=0}^{\infty} a_n\Phi_n(x)r^n$ *converge, for* $0 \leqq r < 1$, *to* $f(x, r)$. *Suppose that*

(i) $|f(x, r)| = o(1/(1-r))$ *uniformly in* x *as* $r \to 1$;

(ii) *there is a function* $y_1 \in \mathscr{H}$ *such that* $-\infty < y_1(x) \leqq f_*(x) \leqq f^*(x) < +\infty$ *for all* x;

(iii) *there is a function* $y_2 \in \mathscr{H}$ *such that* $-\infty < y_2(x) \leqq F(x)$ *for all* x.

Then the series $\sum_{n=0}^{\infty} a_n\Phi_n(x)$ *is Poisson summable almost everywhere, and is the Hermite series of its Poisson sum.*

THEOREM II. *Let the series* $\sum_{n=0}^{\infty} a_n\Phi_n(x)r^n$ *converge, for* $0 \leqq r < 1$, *to* $f(x, r)$. *Suppose that*

(i) $|f(x, r)| = o(1/(1-r))$ *uniformly in* x *as* $r \to 1$;

(ii) $\lim_{r\to 1} f(x, r) = 0$ *for all* x.

Then $a_n = 0$ *for all* n.

In Chapter II we shall show that these theorems are, in a certain sense, best possible. They extend several of the main results in Rudin's paper [2]. This paper was motivated by V. L. Shapiro's work on harmonic analysis and the heat equation, in particular [4].

CHAPTER I

Fundamental lemmas.

LEMMA 1. *Suppose*

(i) $\mathscr{D}$ *is the open rectangle*: $-R < x < R$, $T_1 < t < T_2$ *with boundary* $\dot{\mathscr{D}}$. *Let* $\dot{\mathscr{D}}_1 = \{(x, t) \mid -R < x < R, t = T_2\}$ *and let* $\dot{\mathscr{D}}_2 = \dot{\mathscr{D}} \sim \dot{\mathscr{D}}_1$;

(ii) $h(x, t)$ *satisfies*

(7) $$h_{xx}(x, t) - (x^2+1)h(x, t) = 2h_t(x, t) \quad \text{in } \mathscr{D} \cup \dot{\mathscr{D}}_1,$$

where

$$h(x, t) \in \mathscr{C}^2(\mathscr{D} \cup \dot{\mathscr{D}}_1) \qquad \text{and} \qquad h(x, t) \in \mathscr{C}^0(\mathscr{D} \cup \dot{\mathscr{D}});$$

(iii) $\liminf_{(x,t)\to(x_0,t_0)} h(x, t) \geqq 0$ *for all* $(x_0, t_0) \in \dot{\mathscr{D}}_2$.

Then (a) $h(x, t) \geqq 0$ *in* $\mathscr{D} \cup \dot{\mathscr{D}}$;

(b) $h(x, t)$ *assumes its maximum in* $\dot{\mathscr{D}}_2$.

Proof. Suppose $h(x_1, t_1)<0$, where $(x_1, t_1)\in\mathcal{D}$. By (iii) and the continuity of $h(x,t)$ on the compact set $\mathcal{D}\cup\dot{\mathcal{D}}$, $h(x,t)$ must assume its minimum in $\mathcal{D}\cup\dot{\mathcal{D}}_1$, say at $(x_0, t_0)\in\mathcal{D}\cup\dot{\mathcal{D}}_1$, and $h(x_0,t_0)<0$. This means $h_{xx}(x_0,t_0)\geqq 0$ and $h_t(x_0,t_0)\leqq 0$. Since $h(x,t)$ satisfies (7) in $\dot{\mathcal{D}}\cup\dot{\mathcal{D}}_1$, at (x_0,t_0) we have

$$\underbrace{h_{xx}(x_0,t_0)}_{\geqq 0}\underbrace{-2h_t(x_0,t_0)}_{\geqq 0}\underbrace{-(x_0^2+1)h(x_0,t_0)}_{>0} = 0.$$

This is a contradiction in signs, proving (a).

Suppose $h(x,t)$ assumes its maximum in $\mathcal{D}\cup\dot{\mathcal{D}}_1$, at a point (x_2,t_2). Then $h_t(x_2,t_2)\geqq 0$ and $h_{xx}(x_2,t_2)\leqq 0$. Again using (7) we have

$$\underbrace{h_{xx}(x_2,t_2)}_{\leqq 0}\underbrace{-2h_t(x_2,t_2)}_{\leqq 0}\underbrace{-(x_2^2+1)h(x_2,t_2)}_{\leqq 0} = 0.$$

This means each term must be zero and $h(x_2,t_2)=0$. Since $h(x,t)\geqq 0$ in $\mathcal{D}\cup\dot{\mathcal{D}}$, the maximum in $\mathcal{D}\cup\dot{\mathcal{D}}_1$ is zero. This means $h(x,t)=0$ for $(x,t)\in\mathcal{D}\cup\dot{\mathcal{D}}$, hence in any case $h(x,t)$ assumes its maximum in $\dot{\mathcal{D}}_2$.

LEMMA 2. *Suppose*

(i) $N(x,r)$ *satisfies*

(8) $$N_{xx}(x,r)-(x^2+1)N(x,r)+2[rN(x,r)]_r = 0$$

for $-\infty<x<+\infty$ *and* $0\leqq r<1$, *where* $N(x,r)\in\mathscr{C}^2$ *for* $-\infty<x<+\infty$ *and* $0\leqq r<1$;

(ii) $\lim_{r\to 1} N(x,r)=0$ *uniformly on compact subsets*;

(iii) $|N(x,r)|\leqq K$, *a positive constant for* $-\infty<r<+\infty$ *and* $0\leqq r<1$.

Then $N(x,r)\equiv 0$ *for* $-\infty<x<+\infty$ *and* $0\leqq r<1$.

Proof. Define $h(x,t)=e^{-t}N(x,e^{-t})$ for $-\infty<x<\infty$ and $0<t<\infty$. Then $h(x,t)$ satisfies (7). We note $|h(x,t)|=e^{-t}|N(x,e^{-t})|\leqq K$, a constant for all x and $t>0$. $\lim_{t\to 0} h(x,t)=\lim_{t\to 0} e^{-t}\lim_{t\to 0} N(x,e^{-t})=0$ uniformly on compact subsets. To show $N(x,r)\equiv 0$, it is sufficient to show $h(x,t)\equiv 0$. Consider the function

$$B(x,t) = \frac{1}{(\pi(1-e^{-2t}))^{1/2}}\exp\left\{\frac{x^2-2t}{2}-\frac{x^2}{1-e^{-2t}}\right\} \qquad \text{for } t>0.$$

Claim B satisfies (7) for $t>0$ and all x.

$$B_x(x,t) = \frac{-x}{\pi^{1/2}}\exp\left\{\frac{x^2-2t}{2}-\frac{x^2}{1-e^{-2t}}\right\}\frac{(1+e^{-2t})}{(1-e^{-2t})^{3/2}},$$

$$B_{xx}(x,t) = \frac{-1}{\pi^{1/2}}\frac{\exp\{(x^2-2t)/2-x^2/(1-e^{-2t})\}}{(1-e^{-2t})^{5/2}}[(1-e^{-4t})-x^2(1+e^{-2t})^2],$$

$$B_t(x,t) = \frac{-1}{\pi^{1/2}}\frac{\exp\{(x^2-2t)/2-x^2/(1-e^{-2t})\}}{(1-e^{-2t})^{5/2}}[(1-e^{-2t})-2x^2e^{-2t}].$$

Thus $B_{xx}(x,t)-2B_t(x,t)=(x^2+1)B(x,t)$ which is (7). Set

$$U^R(x,t) = K[B(x+R,t)+B(x-R,t)]$$

for $R>0$ and $t>0$. Then $U^R(x, t)$ satisfies (7) since B does, for all x and $t>0$. Note

$$B(0, t) = e^{-t}/(\pi(1-e^{-2t}))^{1/2} = 1/(\pi(e^{2t}-1))^{1/2} \qquad \text{for } t > 0.$$

Thus $U^R(+R, t) = K[B(2R, t)+B(0, t)] \geqq KB(0, t)$ since $B \geqq 0$ and $U^R(-R, t) = K[B(0, t)+B(-2R, t)] \geqq KB(0, t)$ since $B \geqq 0$. Thus

$$U^R(\pm R, t) \geqq K/(\pi(e^{2t}-1))^{1/2} \qquad \text{for } t > 0.$$

Let T be an arbitrary positive number. For $0<t\leqq T$ we have

$$|h(\pm R, t)| \leqq K \leqq K(\pi(e^{2T}-1))^{1/2}/(\pi(e^{2t}-1))^{1/2} \leqq (\pi(e^{2T}-1))^{1/2}\, U^R(\pm R, t) \quad \text{for } t > 0. \tag{9}$$

Hence the functions $(\pi(e^{2T}-1))^{1/2}\, U^R(x, t) \pm h(x, t)$ which both satisfy (7) in the open rectangle $-R<x<R$ and $0<t<T$ are nonnegative on the sides by (9). Call the open rectangle $D_{R,T}$. On the lower boundary of $D_{R,T}$ we have

$$\lim_{t\to 0} h(x, t) = 0 \quad \text{and} \quad \lim_{t\to 0} U^R(x, t) = 0 \text{ uniformly.}$$

Thus Lemma 1 applies and we have

$$(\pi(e^{2T}-1))^{1/2}\, U^R(x, t) \pm h(x, t) \geqq 0 \text{ in } D_{R,T} \text{ closure.}$$

This means $|h(x, t)| \leqq (\pi(e^{2T}-1))^{1/2}\, U^R(x, t)$. Holding (x, t) fixed and letting $R\to\infty$ we have $\lim_{R\to\infty} U^R(x, t)=0$.

Thus $|h(x, t)|$ can be made arbitrarily small which gives

$$h(x, t) \equiv 0 \quad \text{for } -\infty < x < +\infty \text{ and } 0 < t < T.$$

But T was arbitrary, thus $h(x, t)\equiv 0$ for $-\infty<x<\infty$ and all $t>0$.

LEMMA 3. *Let* $f(x, r)=\lim_{N\to\infty}\sum_{k=0}^{N} a_k\Phi_k(x)r^k$ *exist for* $-\infty<x<+\infty$ *and* $0\leqq r<1$.

Then (a) *for fixed* r, $0\leqq r<1$, $f(x, r)$ *converges uniformly in* x.

(b) $f(x, r)\in\mathscr{H}$ *for* $0\leqq r<1$.

Proof. Since $f(x, r)$ converges, $\lim_{n\to\infty} a_n\Phi_n(x)r^n=0$ for all x and fixed r, $0\leqq r<1$. By [2, Theorem 7],

$$a_n r^n = o(n^{1/4}) \quad \text{for } 0 \leqq r < 1. \tag{10}$$

By choosing δ such that $(1+\delta)r<1$ we see that $a_n[(1+\delta)r]^n \leqq Kn^{1/4}$ for all n, where K is a positive constant. Since $|H_n(x)| < 1.1(n!)^{1/2}\, 2^{n/2} \exp(x^2/2)$ for all x and n [5, p.324], we get

$$|\Phi_n(x)| < 1 \quad \text{for all } x \text{ and } n. \tag{11}$$

Thus

$$|f(x, r)| \leqq \sum_{n=0}^{\infty} |a_n r^n|\, |\Phi_n(x)| \leqq K \sum_{n=0}^{\infty} \frac{n^{1/4}}{(1+\delta)^n} = K_1 < +\infty,$$

by the Weierstrass M-test for $0\leqq r<1$, which proves (a).

Since

$$\int_{-\infty}^{\infty} |x^p f(x, r)| \exp(-x^2/2)\, dx \leqq \int_{-\infty}^{\infty} |x|^p \sum_{n=0}^{\infty} \frac{Kn^{1/4}}{(1+\delta)^n} \exp(-x^2/2)\, dx$$
$$\leqq K_1 \int_{-\infty}^{\infty} |x^p| \exp(-x^2/2)\, dx < +\infty$$

for all $p \geqq 0$, we have (b).

The Poisson kernel for Hermite series [6] has the form

$$(12)\quad P(x, t, r) = \sum_{n=0}^{\infty} \Phi_n(x)\Phi_n(t) r^n = [\pi(1-r^2)]^{1/2} \exp\left\{\frac{x^2-t^2}{2} - \frac{(x-rt)^2}{1-r^2}\right\} \quad (0 \leqq r < 1).$$

For ease in integration we write

$$(13)\quad P(x, t, r) = \frac{\exp\left(-\frac{x^2}{2}\left(\frac{1-r^2}{1+r^2}\right)\right)}{\pi^{1/2}(1-r^2)^{1/2}} \exp\left\{-\frac{(1+r^2)}{2(1-r^2)}\left(t - \frac{2xr}{1+r^2}\right)^2\right\} \quad (0 \leqq r < 1)$$

and

$$(14)\quad P_x(x, t, r) = \left[\frac{-x(1+r^2)+2rt}{1-r^2}\right] P(x, t, r) \qquad (0 \leqq r < 1).$$

LEMMA 4. *Let* $f(x, r) = \sum_{n=0}^{\infty} a_n \Phi_n(x) r^n$ *exist for* $-\infty < x < +\infty$ *and* $0 \leqq r < 1$. *Suppose that* $|f(x, r)| \leqq \varepsilon(r)/(1-r)$ *for* $-\infty < x < +\infty$ *and* $0 \leqq r < 1$, *where* $\varepsilon(r)$ *is bounded and* $\varepsilon(r) = o(1)$ *as* $r \to 1$. *Then*

$$f(x, r+r_0-1) = \int_{-\infty}^{\infty} f(t, r_0) P\left(x, t, \frac{r+r_0-1}{r_0}\right) dt$$

for $0 < r_0 < 1$ *and* $1 - r_0 < r < 1$.

Proof. By [2, p. 398], we can differentiate across the integral and show

$$\int_{-\infty}^{\infty} f(t, r_0) P\left(x, t, \frac{r+r_0-1}{r_0}\right) dt$$

satisfies (8). Using the substitution $z = (r+r_0-1)/r_0$ we have

$$\lim_{r \to 1} \int_{-\infty}^{\infty} f(t, r_0) P\left(x, t, \frac{r+r_0-1}{r_0}\right) dt = \lim_{z \to 1} \int_{-\infty}^{\infty} f(t, r_0) P(x, t, z)\, dt = f(x, r_0)$$

uniformly on compact subsets, since $f(x, r_0)$ is continuous and in $\mathscr{H}$.

Since

$$|f(t, r_0)| \leqq \frac{\varepsilon(r_0)}{1-r_0} \quad \text{and} \quad \int_{-\infty}^{\infty} P(x, t, r)\, dt \leqq 2,$$

we have

$$\left|\int_{-\infty}^{\infty} f(t, r_0) P\left(x, t, \frac{r+r_0-1}{r_0}\right) dt\right| \leqq \frac{\varepsilon(r_0)}{1-r_0} < K$$

for all x, $1 - r_0 \leqq r < 1$.

The above three conditions are also met by $f(x, r+r_0-1)$. In fact,

$$|f(x, r+r_0-1)| \leqq \sum_{n=0}^{\infty} |a_n r_0^n|\, |\Phi_n(x)| \leqq K_1 \sum_{n=0}^{\infty} \frac{n^{1/4}}{(1+\delta)^n} = K_2$$

for all x and r, using (10) and choosing δ as in Lemma 3. $f(x, r+r_0-1)$ satisfies (8) since $\Phi_n(x)$ satisfies [7, p. 105]

$$\Phi_n''(x)-(x^2+1)\Phi_n(x) = -(2n+2)\Phi_n(x) \qquad (n = 0, 1, 2, \ldots).$$

The uniform continuity of $f_x(x, r)$ and $f_{xx}(x, r)$ follows from the M-test.

Thus

$$f(x, r+r_0-1)-\int_{-\infty}^{\infty} f(t, r_0)P\left(x, t, \frac{r+r_0-1}{r_0}\right) dt$$

satisfies all the conditions of Lemma 2, proving this lemma.

LEMMA 5. *Let* $f(x, r)=\sum_{n=0}^{\infty} a_n\Phi_n(x)r^n$ *exist for* $-\infty<x<+\infty$ *and* $0\leqq r<1$. *Suppose that* $|f(x, r)|\leqq \varepsilon(r)/(1-r)$ *for* $-\infty<x<+\infty$ *and* $0\leqq r<1$, *where* $\varepsilon(r)$ *is bounded and* $\varepsilon(r)=o(1)$ *as* $r \to 1$.

Then (a) $f_x(x, r)=\sum_{n=0}^{\infty} a_n\Phi_n'(x)r^n$ *for* $0\leqq r<1$;

(b) *for fixed r*, $0\leqq r<1$, $f_x(x, r)$ *converges uniformly in* x;

(c) $f_x(x, r) \in \mathscr{H}$ *for* $0\leqq r<1$;

(d) $$f_x(x, v+r-1) = \int_{-\infty}^{\infty} f(t, r)P_x\left(x, t, \frac{v+r-1}{r}\right) dt = \sum_{n=0}^{\infty} a_n\Phi_n'(x)(v+r-1)^n$$

for $0<r<1$ *and* $1-r<v<1$;

(e) $|f_x(x, r)|\leqq \varepsilon_1(r)/(1-r)^{3/2}$ *for* $-\infty<x<+\infty$ *and* $\frac{1}{2}\leqq r_0<r<1$, *where* $\varepsilon_1(r)$ *is bounded and* $\varepsilon_1(r)=o(1)$ *as* $r \to 1$.

Proof. By the recursion formula for Hermite polynomials [1, p. 286],

(15) $$2\Phi_n'(x) = (2n)^{1/2}\Phi_{n-1}(x)-(2n+2)^{1/2}\Phi_{n+1}(x) \quad \text{for } n \geqq 1.$$

Using (11) and (15), we have $|\Phi_n'(x)|\leqq An^{1/2}$ for all n and x where A is a positive constant. Thus

$$\left|\sum_{n=0}^{\infty} a_n\Phi_n'(x)r^n\right| \leqq \sum_{n=0}^{\infty} |a_n r^n|\, |\Phi_n'(x)| \leqq K \sum_{n=0}^{\infty} \frac{n^{1/4}An^{1/2}}{(1+\delta)^n} = K_2 < +\infty,$$

using (10) and the ratio test. By the Weierstrass M-test $f_x(x, r)=\sum_{n=0}^{\infty} a_n\Phi_n'(x)r^n$ converges uniformly for all x and $0\leqq r<1$, and the representation is justified, proving (a) and (b).

Since $\int_{-\infty}^{\infty} |x^p f_x(x, r)| \exp(-x^2/2)\, dx\leqq K_2 \int_{-\infty}^{\infty} |x^p| \exp(-x^2/2)\, dx<+\infty$ for all $p\geqq 0$ and $0\leqq r<1$ we have (c).

By Lemma 4

$$f(x, v+r-1) = \int_{-\infty}^{\infty} f(t, r)P\left(x, t, \frac{v+r-1}{r}\right) dt$$

for $0<r<1$ and $1-r<v<1$. Since $f(t,r)\in\mathscr{H}$ we can differentiate across the integral sign [2, pp. 398, 399] to obtain (d).

Set $\rho=(v+r-1)/r$. Using (d), (14) and the substitution

$$y = [(1+\rho^2)/2(1-\rho^2)]^{1/2}(t-2x\rho/(1+\rho^2)),$$

$$|f_x(x, v+r-1)| = \frac{\varepsilon(r)}{1-r}\frac{\exp\left\{-\frac{x^2}{2}\left(\frac{1-\rho^2}{1+\rho^2}\right)\right\}}{\pi^{1/2}(1-\rho^2)^{3/2}}\int_{-\infty}^{\infty}|-x(1+\rho^2)+2t\rho|\exp(-y^2)\,dt$$

$$= \frac{\varepsilon(r)}{1-r}\frac{\exp\left\{-\frac{x^2}{2}\left(\frac{1-\rho^2}{1+\rho^2}\right)\right\}4\rho}{\pi^{1/2}(1-\rho^2)^{1/2}(1+\rho^2)}$$

$$\times\int_{-\infty}^{\infty}\left|y-\frac{x(1-\rho^2)^{3/2}}{[8\rho^2(1+\rho^2)]^{1/2}}\right|\exp(-y^2)\,dy.$$

Set $g_1(x,\rho,r)$ equal to the term outside the integral, and

$$A(x,\rho) = \frac{x(1-\rho^2)^{3/2}}{[8\rho^2(1+\rho^2)]^{1/2}}.$$

Then

$$|f_x(x, v+r-1)| = g_1(x,\rho,r)\left\{\int_{-\infty}^{A}(A-y)\exp(-y^2)\,dy\right.$$

$$\left.+\int_{A}^{\infty}(y-A)\exp(-y^2)\,dy\right\}$$

$$= g_1(x,\rho,r)\left\{A\int_{-A}^{A}\exp(-y^2)\,dy-\int_{A}^{\infty}(-2y)\exp(-y^2)\,dy\right\}$$

$$\leqq g_1(x,\rho,r)\left\{A\int_{-A}^{A}\exp(-y^2)\,dy+1\right\}.$$

Note $g_1(x,\rho,r)\leqq\varepsilon(r)K_3/(1-r)(1-\rho)^{1/2}$, where K_3 is a positive constant. Recalling $\rho=(v+r-1)/r$, set $v=r$ to get $1/(1-\rho)<1/(1-r)$, and $g_1(x,(2r-1)/r,r)=\varepsilon_1(r)/(1-r)^{3/2}$ if $1>\rho>0$. To ensure this, choose $r_0>\frac{1}{2}$. Then, for $r>r_0$, we have $2-1/r>0$, $(2r-1)/r>0$, and $\rho=(2r-1)/r>0$. Thus for $r>r_0$,

$$g_1(x,(2r-1)/r,r) = \varepsilon_1(r)/(1-r)^{3/2},$$

where $\varepsilon_1(r)$ is bounded and $o(1)$ as $r\to 1$.

Part (e) of Lemma 5 will be proven if we show that

$$g_1\left(x,\frac{2r-1}{r},r\right)A\int_{-A}^{A}\exp(-y^2)\,dy \leqq \frac{\varepsilon_1(r)}{(1-r)^{3/2}}$$

or that

$$\frac{\varepsilon(r)}{1-r}K_4\frac{\exp\left\{-\frac{x^2}{2}\left(\frac{1-\rho^2}{1+\rho^2}\right)\right\}}{(1-\rho^2)^{1/2}}|x|(1-\rho^2)^{3/2}\int_{-\infty}^{\infty}\exp(-y^2)\,dy \leqq \frac{\varepsilon_1(r)}{(1-r)^{3/2}}$$

for $r_0 < r < 1$.

It is sufficient to show

$$|x|\left(\frac{1-\rho^2}{1+\rho^2}\right)\exp\left\{-\frac{x^2}{2}\left(\frac{1-\rho^2}{1+\rho^2}\right)\right\}$$

is uniformly bounded for ρ near 1. Setting $\eta=(1-\rho^2)/(1+\rho^2)$ and noting $\eta\to 0$ as $\rho\to 1$ we need only show $|x|\eta\exp((-x^2/2)\eta)$ is uniformly bounded. In fact, for $0\leqq\eta<\frac{1}{2}$, $|x|\eta\exp((-x^2/2)\eta)<1$.

Generalized Hermite operators. $\Phi_n(x)$ satisfies [7, p. 105]

$$\Phi_n''(x)-(x^2+1)\Phi_n(x) = -(2n+2)\Phi_n(x) \qquad (n = 0, 1, 2, \ldots). \tag{16}$$

We consider the equation

$$y''(x)-(x^2+1)y(x) = 0. \tag{17}$$

Putting

$$\beta(x) = \exp(x^2/2)\int_{-\infty}^{x}\exp(-u^2)\,du,$$

we note that $\beta(x)$ and $\beta(-x)$ are linearly independent solutions to (17). Given a function $F(t)$, defined in a neighborhood of the point x, and $h>0$, there exists a unique function $y(t)$ which is a solution of (17) and is such that $y(x+h)=F(x+h)$ and $y(x-h)=F(x-h)$. We define

$$\Lambda F(x) = \lim_{h\to 0}\frac{2[y(x)-F(x)]}{h^2} \tag{18}$$

provided the limit exists. $\Lambda^*F(x)$ and $\Lambda_*F(x)$ are defined likewise with $\limsup_{h\to 0}$ and $\liminf_{h\to 0}$ in place of lim. By [2, p. 388],

$$\Lambda F(x) = F''(x)-(x^2+1)F(x) \quad \text{if } F''(x) \tag{19}$$

exists.

Setting

$$\begin{aligned} k(x, t) &= \pi^{-1/2}\beta(x)\beta(-t) \qquad (x < t),\\ &= \pi^{-1/2}\beta(-x)\beta(t) \qquad (x \geqq t), \end{aligned}$$

we define

$$\Omega f(x) = -\int_{-\infty}^{\infty} f(t)k(x, t)\,dt, \tag{20}$$

provided $f\in\mathscr{H}_0$. The Ω operator is the inverse of Λ. In particular, for $f\in\mathscr{H}$, $f(x)\sim\sum_{n=0}^{\infty}a_n\Phi_n(x)$ iff $\Omega f(x)\sim\sum_{n=0}^{\infty}(a_n/(2n+2))\Phi_n(x)$ [2, p. 389].

LEMMA 6. *Let $f(x, r)=\sum_{n=0}^{\infty}a_n\Phi_n(x)r^n$ exist for $-\infty<x<+\infty$ and $0\leqq r<1$. Suppose that $|f(x, r)|\leqq\varepsilon(r)/(1-r)$ for $-\infty<x<+\infty$ and $0\leqq r<1$, where $\varepsilon(r)$ is bounded and $\varepsilon(r)=o(1)$ as $r\to 1$. Set $F(x, r)=-\sum_{n=0}^{\infty}(a_n/(2n+2))\Phi_n(x)r^n$ for $0\leqq r<1$.*

Then (a) $|F(x, r)| \leqq \varepsilon_2(r) \log (1/(1-r))$ *for* $-\infty < x < +\infty$ *and* $0 \leqq r < 1$, *where* $\varepsilon_2(r)$ *is bounded and* $\varepsilon_2(r) = o(1)$ *as* $r \to 1$;

(b) $F(x, r) = \Omega f(x, r)$ *and* $F(x, r) \in \mathscr{H}$ $(0 \leqq r < 1)$;

(c) $F_x(x, r) = \Omega f_x(x, r) = -\sum_{n=0}^{\infty} (a_n/(2n+2))\Phi_n(x) r^n$ $(0 \leqq r < 1)$;

(d) $|F_x(x, r)| \leqq \varepsilon_3(r)/(1-r)^{1/2}$ *for* $-\infty < x < +\infty$ *and* $\frac{1}{2} \leqq r_0 < r < 1$, *where* $\varepsilon_3(r)$ *is bounded and* $\varepsilon_3(r) = o(1)$ *as* $r \to 1$.

Proof.

$$|rF(x, r) - r_0 F(x, r_0)| = \left| -\sum_{n=0}^{\infty} \frac{a_n}{2n+2} \Phi_n(x) r^{n+1} + \sum_{n=0}^{\infty} \frac{a_n}{2n+2} \Phi_n(x) r_0^{n+1} \right|$$

$$= \left| \sum_{n=0}^{\infty} \frac{a_n}{2} \Phi_n(x) \int_r^{r_0} \rho^n \, d\rho \right| = \left| \int_{r_0}^{r} \frac{1}{2} \sum_{n=0}^{\infty} a_n \Phi_n(x) \rho^n \, d\rho \right|$$

since $f(x, r)$ is a power series in r and converges uniformly over compact subsets of r when r is bounded away from 1.

$$\tfrac{1}{2} \int_{r_0}^{r} |f(x, \rho)| \, d\rho \leqq \tfrac{1}{2} \int_{r_0}^{r} \frac{\varepsilon_1(\rho)}{1-\rho} \, d\rho = \varepsilon_2(r) \log \frac{1}{1-r},$$

where $\varepsilon_2(r)$ is bounded and $o(1)$ as $r \to 1$.

Thus $F(x, r)$ satisfies all the hypotheses of Lemmas 3, 4, and 5 proving $F(x, r) \in \mathscr{H}$, for $0 \leqq r < 1$, $F_x(x, r) \in \mathscr{H}$, and $F_x(x, r) = -\sum_{n=0}^{\infty} (a_n/(2n+2))\Phi_n'(x) r^n$. By [2, Theorems 2 and 8] we have $F(x, r) = \Omega f(x, r)$ and $F_x(x, r) = \Omega f_x(x, r)$.

It remains to prove (d). By Lemma 5, for $r_0 < r < 1$, we have $|f_x(x, r)| \leqq \varepsilon_1(r)/(1-r)^{3/2}$. Thus

$$|rF_x(x, r) - r_0 F_x(x, r_0)| \leqq \left| -\tfrac{1}{2} \sum_{n=0} a_n \Phi_n'(x) \left[\frac{r^{n+1}}{n+1} - \frac{r_0^{n+1}}{n+1} \right] \right|$$

$$\leqq \left| \tfrac{1}{2} \int_{r_0}^{r} \sum_{n=0}^{\infty} a_n \Phi_n'(x) \rho^n \, d\rho \right| \leqq \tfrac{1}{2} \int_{r_0}^{r} |f_x(x, \rho)| \, d\rho$$

$$\leqq \frac{\varepsilon_3(r)}{2(1-r)^{1/2}} \quad \text{for } -\infty < x < +\infty \text{ and } \tfrac{1}{2} \leqq r_0 < r < 1,$$

where $\varepsilon_3(r)$ is bounded and $\varepsilon_3(r) = o(1)$ as $r \to 1$. Thus

$$|F_x(x, r)| \leqq \frac{\varepsilon_3(r)}{(1-r)^{1/2}} \quad \text{for } 0 \leqq r_0 < r < 1,$$

proving (d).

Smoothness. We shall say that G is smooth at the point x if $G(x)$ is defined and finite in a neighborhood of x and if

$$\lim_{h \to 0} [G(x+h) + G(x-h) - 2G(x)]/h = 0. \tag{21}$$

We shall say that G is smooth on $(-\infty, \infty)$ if it is smooth at every point x.

LEMMA 7. *Let* $f(x, r) = \sum_{n=0}^{\infty} a_n \Phi_n(x) r^n$ *exist for* $-\infty < x < +\infty$ *and* $0 \leqq r < 1$. *Suppose that* $|f(x, r)| \leqq \varepsilon(r)/(1-r)$ *for* $-\infty < x < +\infty$ *and* $0 \leqq r < 1$, *where* $\varepsilon(r)$ *is bounded and* $\varepsilon(r) = o(1)$ *as* $r \to 1$.

Set $F(x,r) = -\sum_{n=0}^{\infty} (a_n/(2n+2))\Phi_n(x)r^n$ *for* $0 \leqq r < 1$ *and*

$$G(x,r) = \sum_{n=0}^{\infty} (a_n/(2n+2))^2 \Phi_n'(x) r^n \quad \textit{for } 0 \leqq r < 1.$$

Then

(a) $|G(x,r)| \leqq K_5$, *a positive constant, for all* x *and* $0 \leqq r_1 < r < 1$;

(b) $G(x,r) = \Omega F_x(x,r)$ *and* $G(x,r) \in \mathscr{H}$;

(c) $\lim_{r\to 1} G(x,r) = G(x)$ *uniformly in* x as $r \to 1$, *where* $G(x)$ *is a bounded continuous function on* $(-\infty, \infty)$;

(d) $G(x)$ *is smooth on* $(-\infty, \infty)$.

Proof. Choose r_1 and r such that $1 > r > r_1 > r_0 > \frac{1}{2}$, where r_0 is chosen as in Lemma 6 such that

$$|F_x(x,r)| \leqq \varepsilon_3(r)/(1-r)^{1/2} \quad \text{for } r_0 < r < 1.$$

Then

$$\begin{aligned}
|rG(x,r) - r_1 G(x,r_1)| &= \left| \sum_{n=0}^{\infty} \frac{a_n}{(2n+2)^2} \Phi_n'(x)(r^{n+1} - r_1^{n+1}) \right| \\
&\leqq \tfrac{1}{2} \left| \int_{r_1}^{r} \sum_{n=0}^{\infty} \frac{a_n}{2n+2} \Phi_n'(x) \rho^n \, d\rho \right| \\
&\leqq \tfrac{1}{2} \int_{r_1}^{r} |F_x(x,\rho)| \, d\rho \quad \text{since } F_x(x,r) \text{ converges uniformly} \\
&\leqq \tfrac{1}{2} \int_{r_n}^{r} \frac{\varepsilon_3(\rho)}{(1-\rho)^{1/2}} \, d\rho \leqq K_5[(1-r)^{1/2} - (1-r)^{1/2}].
\end{aligned}$$

Thus, $G(x,r)$ is uniformly bounded for all x and $1 > r > r_0$, and $G(x,r)$ converges uniformly to a bounded continuous function $G(x)$.

Since $G(x,r)$ satisfies the hypothesis of Lemma 6, we have (b).

To prove $G(x)$ is smooth on $(-\infty, \infty)$, let x_0 be given arbitrarily. Select $\delta < (1-r_0)^{1/2}$. Then for $0 < h < \delta$,

$$\begin{aligned}
&[G(x_0+h) + G(x_0-h) - 2G(x_0)]/h \\
&\qquad = [G(x_0+h, 1-h^2) + G(x_0-h, 1-h^2) - 2G(x_0, 1-h^2)]/h \\
&\qquad\quad + [G(x_0+h) - G(x_0+h, 1-h^2)]/h \\
&\qquad\quad + [G(x_0-h) - G(x_0-h, 1-h^2)]/h \\
&\qquad\quad + 2[G(x_0, 1-h^2) - G(x_0)]/h.
\end{aligned}$$

We conclude from (21) that to show G is smooth at x_0, and consequently on $(-\infty, \infty)$, we need only establish

$$\lim_{h\to 0} [G(x, 1-h^2) - G(x)]/h = 0 \quad \text{uniformly for } |x - x_0| \leqq \delta, \tag{22}$$

and

$$G(x_0+h, 1-h^2) + G(x_0-h, 1-h^2) - 2G(x_0, 1-h^2) = o(h) \quad \text{as } h \to 0. \tag{23}$$

To show that (22) holds we will show that given $\varepsilon > 0$ there is an h_0, such that for $0 < h < h_0$ and all x such that $|x - x_0| \leqq \delta$, we have $|G(x, 1-h^2) - G(x)| < h\varepsilon$. Select

r_1 as in Lemma 6 so that for $1>r>r_1>r_0$ we have $|F_x(x, r)|<\varepsilon/2(1-r)^{1/2}$. Choose h_0 such that

(i) $h_0<\delta<(1-r_1)^{1/2}$ and hence $1-h^2>r_1$ for $0<h<h_0$;

(ii) for $0<h<h_0$ we have

$$|G(x, 1-h^2)| < |G(x)|+1 < \max_{|x-x_0|\leqq\delta} |G(x)|+1 = K_6,$$

a positive constant, by continuity of $G(x)$ and the uniform convergence of $G(x, r)$ to $G(x)$;

(iii) $h_0<\varepsilon/2K_6$;

(iv) $h_0<1/(x_0^2+1)$.

Now $|G(x, 1-h^2)-G(x)|\leqq|G(x)-(1-h^2)G(x, 1-h^2)|+h^2|G(x, 1-h^2)|$.

$$\begin{aligned}|G(x)-(1-h^2)G(x, 1-h^2)| &= \left|\lim_{r\to 1} rG(x, r)-\sum_{n=0}^{\infty}(a_n/(2n+2)^2)\Phi_n'(x)(1-h^2)^{n+1}\right| \\ &= \left|\lim_{r\to 1}\sum_{n=0}^{\infty}\frac{a_n}{2(2n+2)}\Phi_n'(x)\left[\frac{r^{n+1}}{n+1}-\frac{(1-h^2)^{n+1}}{n+1}\right]\right| \\ &= \left|\lim_{r\to 1}\tfrac{1}{2}\int_{1-h^2}^{r}\sum_{n=0}^{\infty}\frac{a_n}{2n+2}\Phi_n'(x)\rho^n\,d\rho\right| \\ &\leqq \lim_{r\to 1}\tfrac{1}{2}\int_{1-h^2}^{r}\frac{\varepsilon\,d\rho}{2(1-\rho)^{1/2}} \quad \text{since } 1-h^2 > r_1, \text{ for } r > r_1 \\ &\leqq \lim_{r\to 1}\left[\frac{\varepsilon}{2}h-(1-r)^{1/2}\frac{\varepsilon}{2}\right] = \frac{\varepsilon}{2}h.\end{aligned}$$

Thus

$$\begin{aligned}|G(x, 1-h^2)-G(x)| &\leqq \varepsilon h+h^2|G(x, 1-h^2)|; \\ &\leqq \varepsilon/2h+h^2K_6 \quad \text{for } 0 < h < h_0; \\ &\leqq \varepsilon/2h+h\varepsilon/2 = \varepsilon h, \quad \text{proving (22).}\end{aligned}$$

To prove (23) we will prove, for $\varepsilon>0$ and $0<h<h_0$, that $(1-h^2)|\nabla G|<\varepsilon h$, where $\nabla G=G(x_0+h, 1-h^2)+G(x_0-h, 1-h^2)-2G(x_0, 1-h^2)$.

$$\begin{aligned}(1-h^2)|\nabla G| &= \Bigg|\sum_{n=0}^{\infty}\frac{a_n}{(2n+2)^2}\Phi_n'(x_0+h)(1-h^2)^{n+1} \\ &\quad +\sum_{h=0}^{\infty}\frac{a_n}{(2n+2)^2}\Phi_n'(x_0-h)(1-h^2)^{n+1}-2\sum_{h=0}^{\infty}\frac{a_n}{(2n+2)^2}\Phi_n'(x_0)(1-h^2)^{n+1}\Bigg| \\ &\leqq \tfrac{1}{2}\Bigg|\sum_{n=0}^{\infty}\Bigg\{\frac{a_n}{2n+2}\Phi_n'(x_0+h)\int_{r_0}^{1-h^2}\rho^n\,d\rho+\frac{a_n}{2n+2}\Phi_n'(x_0-h)\int_{r_0}^{1-h^2}\rho^n\,d\rho \\ &\qquad -\frac{2a_n}{2n+2}\Phi_n'(x_0)\int_{r_0}^{1-h^2}\rho^n\,d\rho\Bigg\}\Bigg| \\ &\quad +\left|\frac{a_0}{4}[\Phi_0'(x_0+h)+\Phi_0'(x_0-h)-2\Phi_0'(x_0)]\right| \\ &\leqq \tfrac{1}{2}\left|\int_{r_0}^{1-h^2}[F_x(x_0+h, \rho)+F_x(x_0-h, \rho)-2F_x(x_0, \rho)]\,d\rho\right|+o(h) \\ &\qquad \text{since } \Phi_0' \text{ is smooth} \\ &\leqq \tfrac{1}{2}\left|\int_0^{1-h^2}\int_0^h[F_{xx}(x_0+y, \rho)-F_{xx}(x_0-y, \rho)]\,dy\,d\rho\right|+o(h)\end{aligned}$$

by the uniform convergence of F_x and F_{xx}. Using the fact that $F_{xx}(x_0, \rho) = (x_0^2+1)F(x_0, \rho) - 2\partial[\rho F(x_0, \rho)]/\partial\rho$ we have

$$(1-h^2)|\nabla G| \leqq o(h)+\tfrac{1}{2}\left|\int_0^{1-h^2}\int_0^h \left\{-2\frac{\partial}{\partial\rho}[\rho F(x_0+y,\rho)]+[(x_0+y)^2+1]F(x_0+y,\rho) + 2\frac{\partial}{\partial\rho}[\rho F(x_0-y,\rho)] + [(x_0-y)^2+1]F(x_0-y,\rho)\right\} dy\, d\rho\right|.$$

By Fubini's Theorem, we have

$$\begin{aligned}(1-h^2)|\nabla G| \leqq o(h)+\tfrac{1}{2}\Bigg|&\int_0^h\int_0^{1-h^2}\left\{-2\frac{\partial}{\partial\rho}[\rho F(x_0+y,\rho)-\rho F(x_0-y,\rho)]\,d\rho\,dy\right\} \\ &+\int_0^h\int_0^{1-h^2}\{[(x_0+y)^2+1]F(x_0+y,\rho) \\ &-[(x_0-y)^2+1]F(x_0-y,\rho)\}\,d\rho\,dy\Bigg|. \end{aligned} \tag{24}$$

The first integral in (24) is then

$$\begin{aligned}&\left|\int_0^h(-2)[\rho F(x_0+y,\rho)-\rho F(x_0-y,\rho)]_0^{1-h^2}\,dy\right| \\ &= \left|-2\int_0^h(1-h^2)[F(x_0+y,1-h^2)-F(x_0-y,1-h^2)]\,dy\right| \\ &= \left|-2\int_0^h(1-h^2)\int_{x_0-y}^{x_0+y}F_x(x,1-h^2)\,dx\,dy\right| \\ &\leqq 2\int_0^h(1-h^2)\int_{x_0-y}^{x_0+y}|F_x(x,1-h^2)|\,dx\,dy \\ &\leqq 2\int_0^h\int_{x_0-y}^{x_0+y}(1-h^2)\frac{\varepsilon}{2(1-(1-h^2))^{1/2}}\,dx\,dy \quad \text{since } r_1 < 1-h^2 \\ &\leqq 2\int_0^h\frac{\varepsilon(1-h^2)}{2h}2y\,dy = \frac{\varepsilon}{h}(1-h^2)h^2 < \varepsilon h.\end{aligned}$$

The second integral in (24) is

$$\begin{aligned}&\tfrac{1}{2}\left|\int_0^h\int_0^{1-h^2}\{[(x_0+y)^2+1]F(x_0+y,\rho)-[(x_0-y)^2+1]F(x_0-y,\rho)\}\,d\rho\,dy\right| \\ &\leqq \tfrac{1}{2}(x_0^2+1)\left|\int_0^h\int_0^{1-h^2}[F(x_0+y,\rho)-F(x_0-y,\rho)]\,d\rho\,dy\right| \\ &\quad+\left|\int_0^h y\int_0^{1-h^2}[F(x_0+y,\rho)+F(x_0-y,\rho)]\,d\rho\,dy\right| \\ &\quad+\tfrac{1}{2}\left|\int_0^h y^2\int_0^{1-h^2}[F(x_0+y,\rho)-F(x_0-y,\rho)]\,d\rho\,dy\right| \\ &\leqq \tfrac{1}{2}(x_0^2+1)\Bigg|\int_0^h\int_0^{1-h^2}\int_{-y}^{y}F_t(x_0+t,\rho)\,dt\,d\rho\,dy \\ &\quad+\int_0^h y\int_0^{1-h^2}|F(x_0+y,\rho)+F(x_0-y,\rho)|\,d\rho\,dy \\ &\quad+\tfrac{1}{2}\int_0^h y^2\int_0^{1-h^2}\int_{-y}^{y}F_t(x_0+t,\rho)\,dt\,d\rho\,dy\Bigg|.\end{aligned}$$

Now using the fact that $|F_t(x, r)| < \varepsilon/(1-r)^{1/2}$ for $r > r_1$, $|F_t(x, r)| < K_7$ for $0 \leqq r < r_1$,

$$|F(x, r)| < \varepsilon \log 1/(1-r) \quad \text{for } r > r_2, \qquad |F(x, r)| < K_8 \quad \text{for } 0 \leqq r < r_2,$$

we obtain that the second integral in (24) is bounded by

$$\begin{aligned}
&\tfrac{1}{2}(x_0^2+1) \int_0^h \left[\int_0^{r_1} \int_{-y}^{y} K_7\, dt\, d\rho + \int_{r_1}^{1-h^2} \int_{-y}^{y} \frac{\varepsilon}{(1-\rho)^{1/2}}\, dt\, d\rho\right] dy \\
&\quad + \int_0^h y \int_0^{r_2} 2K_8\, d\rho\, dy + \varepsilon \int_0^h y \int_{r_2}^{1-h^2} \log(1/(1-\rho))\, d\rho\, dy \\
&\quad + \tfrac{1}{2} \int_0^h y^2 \left[\int_0^{r_1} \int_{-y}^{y} K_7\, dt\, d\rho + \int_{r_1}^{1-h^2} \int_{-y}^{y} \frac{\varepsilon}{(1-\rho)^{1/2}}\, dt\, d\rho\right] dy \\
&\leqq \tfrac{1}{2}(x_0^2+1) \int_0^h \left[\int_0^{r_1} K_7\, 2y\, d\rho + \int_{r_1}^{1-h^2} \frac{2\varepsilon y}{(1-\rho)^{1/2}}\, d\rho\right] dy \\
&\quad + 2K_8 \int_0^h y r_2\, dy + \varepsilon \int_0^h y \int_{1-r_2}^{h^2} \log s\, ds\, dy \\
&\quad + \tfrac{1}{2} K_7 \int_0^h y^2 \left[\int_0^{r_1} 2y\, d\rho + \int_{r_1}^{1-h^2} \frac{2y\varepsilon}{(1-\rho)^{1/2}}\, d\rho\right] dy \\
&\leqq \tfrac{1}{2}(x_0^2+1) \int_0^h [2K_7 y r_1 - \varepsilon y (1-\rho)^{1/2}|_{r_1}^{1-h^2}]\, dy \\
&\quad + 2K_8 r_2 \frac{y^2}{2}\bigg|_0^h + \varepsilon \int_0^h y[s \log s - s]_{1-r_2}^{h^2}\, dy \\
&\quad + K_7 r_1 \int_0^h y^3\, dy - K_7 \varepsilon \int_0^h y^3 (1-\rho)^{1/2}\bigg|_{r_1}^{1-h^2} dy \\
&\leqq \tfrac{1}{2}(x_0^2+1)[K_7 r_1 h^2 - 2\varepsilon(h-(1-r_1)^{1/2})h^2] \\
&\quad + K_8 r_2 h^2 + \varepsilon h^2[(h^2 \log h^2 - h^2) - (1-r_2)\log(1-r_2) + (1-r_2)] \\
&\quad + K_7 r_1 (h^4/4) - K_7 \varepsilon [h-(1-r_1)^{1/2}] h^4/4 = +O(h^2) = o(h).
\end{aligned}$$

LEMMA 8. *Let $f(x, r) = \sum_{n=0}^{\infty} a_n \Phi_n(x) r^n$ exist for $-\infty < x < +\infty$ and $0 \leqq r < 1$. Suppose that $|f(x, r)| \leqq \varepsilon(r)/(1-r)$ for $-\infty < x < +\infty$ and $0 \leqq r < 1$, where $\varepsilon(r)$ is bounded and $\varepsilon(r) = o(1)$ as $r \to 1$. Set $F(x, r) = -\sum_{n=0}^{\infty} (a_n/(2n+2))\Phi_n(x) r^n$ for $0 \leqq r < 1$, and suppose that there is $y_1 \in \mathscr{H}$ such that $-\infty < y_1(x) \leqq f_*(x) \leqq f^*(x) < +\infty$ for all x. Then $\lim_{r \to 1} F(x, r) = F(x)$ exists and is finite for all x.*

Proof. It is sufficient to show $\lim_{r\to 1} rF(x, r)$ exists and is finite.

$$rF(x_0, r) = -\sum_{n=0}^{\infty} \frac{a_n}{2n+2} \Phi_n(x_0) r^{n+1} = -\tfrac{1}{2} \int_0^r f(x_0, \rho)\, d\rho.$$

By hypothesis $f_*(x_0)$ and $f^*(x_0)$ are finite, hence $f(x_0, r)$ is bounded in absolute value by $A(x_0) = \sup\{|f^*(x_0)|+1, |f_*(x_0)|+1\}$ for $0 < r_3 < r < 1$. If r_4 and r_5 are such that $0 < r_3 < r_4 < r_5 < 1$, then

$$|r_5 F(x_0, r_5) - r_4 F(x_0, r_4)| \leqq \tfrac{1}{2} \int_{r_4}^{r_5} |f(x_0, \rho)|\, d\rho \leqq \frac{A(x_0)}{2}(r_5 - r_4)$$

which approaches zero as r_4 and r_5 approach 1.

Generalized derivatives. If G is defined and finite in a neighborhood of the point x, we set $D_1^*G(x) = \limsup_{h\to 0} [G(x+h)-G(x-h)]/2h$. $D_{1*}G(x)$ will designate the corresponding $\liminf_{h\to 0}$.

LEMMA 9. *Under the same hypothesis as Lemma* 8, *we have*

$$D_{1*}G(x) \leqq \liminf_{r\to 1} [F(x,r)+(x^2+1)\Omega F(x,r)]$$
$$\leqq \limsup_{r\to 1} [F(x,r)+(x^2+1)\Omega F(x,r)] \leqq D_1^*G(x).$$

Proof. $\Omega F(x,r)$ is well defined since $F(x,r)\in\mathscr{H}$. By Lemmas 7 and 4,

$$G(x, v+r-1) = \int_{-\infty}^{\infty} G(t,r)P\left(x, t, \frac{v+r-1}{r}\right) dt \text{ for } 1 > r > r_0,\ 1 > v+r-1 > r_0.$$

Taking limits as $r\to 1$,

$$G(x,v) = \int_{-\infty}^{\infty} G(t)P(x,t,v)\,dt \quad \text{for } 1 > v > r_0.$$

It is sufficient to show

$$D_{1*}G(x_0) = \liminf_{h\to 0}\frac{G(x_0+h)-G(x_0-h)}{2h} \leqq \liminf_{r\to 1}[F(x_0,r)+(x_0^2+1)\Omega F(x_0,r)].$$

If $D_{1*}G(x_0) = -\infty$ we are done. Assume $D_{1*}G(x_0) > q > -\infty$. Observe

$$G_x(x_0,r) = \sum_{n=0}^{\infty}\frac{a_n}{(2n+2)^2}\Phi_n''(x_0)r^n$$
$$= -\sum_{n=0}^{\infty}\frac{a_n}{2n+2}\Phi_n(x_0)r^n + (x_0^2+1)\sum_{n=0}^{\infty}\frac{a_n}{(2n+2)^2}\Phi_n(x_0)r^n$$

since $\Phi_n(x)$ satisfies (16). Therefore

$$G_x(x_0,r) = F(x_0,r)+(x_0^2+1)\Omega F(x_0,r) = \int_{-\infty}^{\infty} G(t)P_x(x_0,t,r)\,dt,$$

and we have to show

$$D_{1*}G(x_0) \leqq \liminf_{r\to 1}\int_{-\infty}^{\infty} G(t)P_x(x_0,t,r)\,dt = \liminf_{r\to 1} G_x(x_0,r).$$

Since $D_{1*}G(x_0) > q > -\infty$ it is sufficient to show

$$\liminf_{r\to 1} G_x(x_0,r) \geqq q. \tag{25}$$

To prove (25) note that

$$G_x(x_0,r) = \int_{-\infty}^{\infty} G(t)P_x(x_0,t,r)\,dt \qquad \text{which by (14)}$$
$$= \frac{\exp\left\{-\frac{x_0^2}{2}\left(\frac{1-r^2}{1+r^2}\right)\right\}}{\pi^{1/2}(1-r^2)^{3/2}}\int_{-\infty}^{\infty} G(t)[-x_0(1+r^2)+2tr]e^{-u^2}\,dt$$

where

$$u = \left[\frac{1+r^2}{2(1-r^2)}\right]^{1/2}\left(t-\frac{2x_0r}{1+r^2}\right).$$

Rearranging we find

$$G_x(x_0, r) = \frac{\exp\left\{-\frac{x_0^2}{2}\left(\frac{1-r^2}{1+r^2}\right)\right\}}{\pi^{1/2}(1-r^2)^{3/2}}(-x_0)(1-r)^2$$

$$\text{(26)} \qquad \times\left[\int_0^\infty G(x_0+s)\exp\left\{-\left[\frac{1+r^2}{2(1-r^2)}\right]\left[s+x_0\frac{(1-r)^2}{(1+r^2)}\right]^2\right\}ds\right.$$

$$\left.+\int_0^\infty G(x_0-s)\exp\left\{-\left[\frac{1+r^2}{2(1-r^2)}\right]\left[s-x_0\frac{(1-r)^2}{(1+r^2)}\right]^2\right\}ds\right]$$

$$+\frac{\exp\left\{-\frac{x_0^2}{2}\left(\frac{1-r^2}{1+r^2}\right)\right\}2r}{\pi^{1/2}(1-r^2)^{3/2}}$$

$$\times\left[\int_0^\infty G(x_0+s)s\exp\left\{-\left[\frac{1+r^2}{2(1-r^2)}\right]\left[s+x_0\frac{(1-r)^2}{(1+r^2)}\right]^2\right\}ds\right.$$

$$\left.-\int_0^\infty G(x_0-s)s\exp\left\{-\left[\frac{1+r^2}{2(1-r^2)}\right]\left[s-x_0\frac{(1-r)^2}{(1+r^2)}\right]^2\right\}ds\right].$$

The claim is that (26) is $o(1)$ as $r \to 1$. Because $G(x)$ is bounded by K_5 and using the substitutions $t=[(1+r^2)/2(1-r^2)]^{1/2}[s\pm x_0(1-r)^2/(1+r^2)]$, we find

$$\lim_{r\to 1}|(26)| \leqq \frac{|x_0|}{2\pi^{1/2}}K_5\lim_{r\to 1}(1-r)^{1/2}\left(\frac{1-r^2}{1+r^2}\right)^{1/2}\left\{2\int_{-\infty}^\infty e^{-t^2}\,dt\right\} = 0.$$

Therefore $G_x(x_0, r)$ can be written as

$$o(1)+\frac{2r\exp\left\{-\frac{x_0^2}{2}\left(\frac{1-r^2}{1+r^2}\right)\right\}}{\pi^{1/2}(1-r^2)^{3/2}}$$

$$\times\left[\int_0^\infty \{G(x_0+s)-G(x_0-s)\}s\exp\left\{-\left[\frac{1+r^2}{2(1-r^2)}\right]\left[s+x_0\frac{(1-r)^2}{(1+r^2)}\right]^2\right\}ds\right]$$

$$\text{(27)} \qquad +\frac{2r\exp\left\{-\frac{x_0^2}{2}\left(\frac{1-r^2}{1+r^2}\right)\right\}}{\pi^{1/2}(1-r^2)^{3/2}}$$

$$\times\left[\int_0^\infty G(x_0-s)s\exp\left\{-\left[\frac{1+r^2}{2(1-r^2)}\right]\left[s+x_0\frac{(1-r)^2}{(1+r^2)}\right]^2\right\}ds\right.$$

$$\left.-\int_0^\infty G(x_0-s)s\exp\left\{-\left[\frac{1+r^2}{2(1-r^2)}\right]\left[s-x_0\frac{(1-r)^2}{(1+r^2)}\right]^2\right\}ds\right].$$

Using the same substitutions, we get $\lim_{r\to 1}|(27)|=0$, since the exponential term dominates. Thus

$$G_x(x_0, r) = o(1)+\frac{2r\exp\left\{-\frac{x_0^2}{2}\left(\frac{1-r^2}{1+r^2}\right)\right\}}{\pi^{1/2}(1-r^2)^{3/2}}$$

$$\times\left[\int_0^\infty \{G(x_0+s)-G(x_0-s)\}s\exp\left\{-\left[\frac{1+r^2}{2(1-r^2)}\right]\left[s+x_0\frac{(1-r)^2}{(1+r^2)}\right]^2\right\}ds\right].$$

Note it is sufficient to consider the integral only over the interval $s \in [0, \delta]$, for any fixed $\delta > 0$, because the exponential term carries the integral to zero as $r \to 1$, when s is bounded away from zero. Therefore

$$G_x(x_0, r) = o(1) + \frac{2r \exp\left\{-\frac{x_0^2}{2}\left(\frac{1-r^2}{1+r^2}\right)\right\}}{\pi^{1/2}(1-r^2)^{3/2}} \tag{28}$$
$$\times \left[\int_0^\delta \{G(x_0+s) - G(x_0-s)\} s \exp\left\{-\left[\frac{1+r^2}{2(1-r^2)}\right]\left[s + x_0\frac{(1-r)^2}{(1+r^2)}\right]^2\right\} ds\right].$$

Since $D_{1*}G(x_0) > q$, we choose δ such that, for $|s| < \delta$, we have

$$\{G(x_0+s) - G(x_0-s)\} > 2qs.$$

Then (28) dominates

$$\frac{\exp\left\{-\frac{x_0^2}{2}\left(\frac{1-r^2}{1+r^2}\right)\right\} 2r2q}{\pi^{1/2}(1-r^2)^{3/2}} \left[\int_0^\delta s^2 \exp\left\{-\left[\frac{1+r^2}{2(1-r^2)}\right]\left[s + x_0\frac{(1-r)^2}{(1+r^2)}\right]^2\right\} ds\right].$$

By the usual substitution, and letting $r \to 1$,

$$\liminf_{r\to 1} G_x(x_0, r) \geqq \frac{4q}{\pi^{1/2}} \int_0^\infty t^2 e^{-t^2}\, dt = q.$$

LEMMA 10. *Let $f(x, r) = \sum_{n=0}^{\infty} a_n \Phi_n(x) r^n$ exist for $-\infty < x < +\infty$ and $0 \leqq r < 1$. Suppose that $|f(x, r)| \leqq \varepsilon(r)/(1-r)$ for $-\infty < x < +\infty$ and $0 \leqq r < 1$, where $\varepsilon(r)$ is bounded and $\varepsilon(r) = o(1)$ as $r \to 1$, and that there is $y_1 \in \mathscr{H}$ such that $-\infty < y_1 \leqq f_*(x) \leqq f^*(x) < +\infty$ for all x. Set $F(x, r) = -\sum_{n=0}^{\infty} (a_n/(2n+2))\Phi_n(x) r^n$ for $0 \leqq r < 1$, and suppose $F(x, r) \to F(x)$, where $F(x)$ is a continuous function on (a, b). Then*

(a) *$F(x, r) \to F(x)$ uniformly on compact subsets of (a, b) as $r \to 1$;*

(b) *$\Lambda^* F(x) \geqq f_*(x)$ and $f^*(x) \geqq \Lambda_* F(x)$ in (a, b).*

Proof. Let $[a_1, b_1] \subset (a, b)$. We wish to show $F(x, r) \to F(x)$ uniformly in $[a_1, b_1]$ as $r \to 1$. Let

$$a < a_4 < a_3 < a_2 < a_1 < b_1 < b_2 < b_3 < b_4 < b. \tag{29}$$

Let $\lambda(x)$ be a $\mathscr{C}^\infty$ function such that

$$\begin{aligned} \lambda(x) &= 1 \quad \text{in } [a_3, b_3], \\ \lambda(x) &= 0 \quad \text{for } x \notin [a_4, b_4], \\ \lambda(x) &\leqq 1. \end{aligned} \tag{30}$$

$$\begin{aligned} &\text{For } x \in (a, b) \text{ define } F_1(x) = \lambda(x)F(x) \\ &\text{and for } x \notin (a, b) \text{ define } F_1(x) = 0. \end{aligned} \tag{31}$$

Then by (29), (30), (31), and the hypothesis we have $F_1(x)$ is continuous, in $\mathscr{L}^1$, $\mathscr{H}$ and bounded by a constant, say K. Set

$$F_1(x, r) = \int_{-\infty}^{\infty} P(x, t, r) F_1(t)\, dt \quad \text{for } 0 \leqq r < 1. \tag{32}$$

Then $F_1(x, r)$ satisfies (8) for $0 \leqq r < 1$. Since $F_1(x)$ is continuous and in $\mathscr{H}$, we have $F_1(x, r) \to F_1(x)$ as $r \to 1$ [2, pp. 398–399]. We will show

(33) $F_1(x, r) \to F_1(x)$ uniformly on compact subsets of $(-\infty, \infty)$ as $r \to 1$. Let $[c, d]$ be any closed interval in $(-\infty, \infty)$. For $\varepsilon > 0$ choose r' such that when $r > r'$,

$$\left| \int_{-\infty}^{\infty} P(x, t, r)\, dt - 1 \right| < \frac{\varepsilon}{2K} \quad \text{on } [c, d].$$

This can be done since $\int_{-\infty}^{\infty} P(x, t, r)\, dt \to 1$ uniformly on compact subsets. Hence for $r > r'$ we have

$$\begin{aligned} |F_1(x, r) - F_1(x)| &\leqq \left| \int_{-\infty}^{\infty} F_1(t) P(x, t, r)\, dt - \int_{-\infty}^{\infty} F_1(x) P(x, t, r)\, dt \right| \\ &\quad + \left| F_1(x) \int_{-\infty}^{\infty} P(x, t, r)\, dt - F_1(x) \right| \\ &\leqq \int_{-\infty}^{\infty} |F_1(t) - F_1(x)| P(x, t, r)\, dt + K \left| \int_{-\infty}^{\infty} P(x, t, r)\, dt - 1 \right| \\ &\leqq \frac{\varepsilon}{2} + \int_{-\infty}^{x-\delta} + \int_{x-\delta}^{x} + \int_{x}^{x+\delta} + \int_{x+\delta}^{\infty} |F_1(t) - F_1(x)| P(x, t, r)\, dt \end{aligned}$$

where $\delta > 0$ is to be chosen. Since $\lim_{r \to 1} P(x, t, r) = 0$ uniformly for $|x - t| \geqq \delta$ and $|F_1(t) - F_1(x)| \leqq 2K$, the integrals $\int_{-\infty}^{x-\delta}$ and $\int_{x+\delta}^{\infty}$ go to zero uniformly on $[c, d]$. Choose δ by the uniform continuity of $F_1(x)$ on $[c, d]$ such that, for the given $\varepsilon > 0$, we have that when $|x - t| < \delta$ then $|F_1(t) - F_1(x)| < \varepsilon/4$. Then

$$\int_{x}^{x+\delta} |F_1(t) - F_1(x)| P(x, t, r)\, dt \leqq \frac{\varepsilon}{4} \int_{x}^{x+\delta} P(x, t, r)\, dt \leqq \frac{\varepsilon}{4} \int_{-\infty}^{\infty} P(x, t, r)\, dt$$

which converges to $\varepsilon/4$ uniformly on compact subsets. The integral $\int_{x-\delta}^{x}$ similarly becomes smaller than $\varepsilon/4$, proving (33).

Since $F_1(x) = F(x)$ on $[a_1, b_1]$, we have $F_1(x, r) \to F(x)$ uniformly on $[a_1, b_1]$. It remains to show $F(x, r) - F_1(x, r) \to 0$ uniformly on $[a_1, b_1]$.

$$H(x, r) = \sum_{n=0}^{\infty} \frac{a_n}{(2n+2)^2} \Phi_n(x) r^n \qquad (0 \leqq r < 1).$$

Claim:

(34) $H(x, r) \to H(x)$, a bounded continuous function, and the convergence is uniform in x.

To see this, choose $r_1 > r_2 > 1/2$. Then

$$\begin{aligned} |r_1 H(x, r_1) - r_2 H(x, r_2)| &\leqq \tfrac{1}{2} \left| \sum_{n=0}^{\infty} \frac{a_n}{2n+2} \Phi_n(x) \int_{r_2}^{r_1} \rho^n\, d\rho \right| \\ &\leqq \tfrac{1}{2} \int_{r_2}^{r_1} |F(x, \rho)|\, d\rho \to 0 \end{aligned}$$

uniformly in x as $r_1, r_2 \to 1$ since $|F(x, r)| \leqq \varepsilon_2(r) \log 1/(1-r)$ by Lemma 6. Thus $H(x, r)$ is bounded for all x and is in $\mathscr{H}$. By [2, Theorems 2 and 8] we also have

$$H(x, r) = \Omega F(x, r) \sim \sum_{n=0}^{\infty} \frac{a_n}{(2n+2)^2} \Phi_n(x) r^n.$$

Observe $H(x, r) - \int_{-\infty}^{\infty} P(x, t, r) H(t)\, dt$ satisfies all the hypothesis of Lemma 2, which proves $H(x, r) = \int_{-\infty}^{\infty} P(x, t, r) H(t)\, dt$. Define

$$H_1(x, r) = \Omega F_1(x, r) \qquad (0 \leqq r < 1).$$

(35) Note $H_1(x, r) \to H_1(x)$ uniformly for x on compact subsets of $(-\infty, \infty)$ where $H_1(x)$ is a bounded and continuous function, since

$$\begin{aligned} \lim_{r\to 1} \Omega F_1(x, r) &= -\lim_{r\to 1} \int_{-\infty}^{\infty} F_1(t, r) k(x, t)\, dt \\ &= -\int_{-\infty}^{\infty} F_1(t) k(x, t)\, dt = \Omega F_1(x) \end{aligned}$$

and $F_1(x, r) \to F_1(x)$ uniformly for x on compact subsets. By Lemma 2 applied to $H_1(x, r) - \int_{-\infty}^{\infty} P(x, t, r) H_1(t)\, dt$ we find

$$H_1(x, r) = \int_{-\infty}^{\infty} P(x, t, r) H_1(t)\, dt \qquad (0 \leqq r < 1).$$

Set

$$H_2(x, r) = H(x, r) - H_1(x, r) \qquad (0 \leqq r < 1).$$

Then by (34) and (35), setting $H_2(x) = H(x) - H_1(x)$ we have $H_2(x, r) \to H_2(x)$ uniformly for x in compact subsets, where $H_2(x)$ is a bounded continuous function in $\mathscr{H}$.

Set

$$F_2(x, r) = F(x, r) - F_1(x, r) \qquad (0 \leqq r < 1).$$

(36) Then $F_2(x, r) \to F_2(x)$, where $F_2(x) = 0$ for $x \in [a_3, b_3]$.

Setting

$$F_1(x, r) \sim -\sum_{n=0}^{\infty} \frac{a_n'}{2n+2} \Phi_n(x) r^n,$$

we have

$$H_1(x, r) = \Omega F_1(x, r) \sim \sum_{n=0}^{\infty} \frac{a_n'}{(2n+2)^2} \Phi_n(x) r^n$$

and

$$H_2(x, r) \sim \sum_{n=0}^{\infty} \frac{(a_n - a_n')}{(2n+2)^2} \Phi_n(x) r^n = \sum_{n=0}^{\infty} \frac{\tilde{a}_n}{(2n+2)^2} \Phi_n(x) r^n.$$

Claim:

$$H_2(x) \sim \sum_{n=0}^{\infty} \frac{\tilde{a}_n}{(2n+2)^2} \Phi_n(x). \tag{37}$$

Since $H_2(x) \in \mathscr{H}$ we have

$$H_2(x) \sim \sum_{n=0}^{\infty} \frac{b_n}{(2n+2)^2} \Phi_n(x)$$

and by the continuity of $H_2(x)$,

$$\sum_{n=0}^{\infty} \frac{b_n}{(2n+2)^2} \Phi_n(x) r^n \to H_2(x)$$

uniformly on compact subsets. We also have

$$\sum_{n=0}^{\infty} \frac{\tilde{a}_n}{(2n+2)^2} \Phi_n(x) r^n \to H_2(x)$$

uniformly on compact subsets. Thus applying Lemma 2 to

$$\sum ((b_n - \tilde{a}_n)/(2n+2)^2)\Phi_n(x) r^n$$

we have

$$\tilde{a}_n = b_n \quad \text{for all } n, \text{ proving (37).}$$

By [2, Theorem 4], $\Lambda_* H_2(x) \leqq F_2^*(x)$ and $F_{2*}(x) \leqq \Lambda^* H_2(x)$. By (36) we have $\Lambda_* H_2(x) \leqq 0 \leqq \Lambda^* H_2(x)$ on $[a_3, b_3]$. By [2, p. 395], $H_2(x)$ satisfies (17) in $[a_3, b_3]$.

(38) That is, $H_2(x)$ is twice differentiable in $[a_3, b_3]$ and $\Lambda H_2(x) = 0$ in $[a_3, b_3]$. Now $\Lambda H_2(x, r) = \int_{-\infty}^{\infty} \Lambda P(x, t, r) H_2(t)\, dt$ since $H_2(x) \in \mathscr{H}$ by [2, p. 398]. We will show

$$\text{(39)} \qquad \Lambda H_2(x, r) \to 0 \quad \text{uniformly in } [a_1, b_1] \text{ as } r \to 1.$$

Now $\Lambda P(x, t, r) = -2[rP(x, t, r)]_r = P_{tt}(x, t, r) - (t^2+1)P(x, t, r)$. Since $H_2(t)$ is bounded, $\int_{x+\delta}^{\infty} \Lambda P(x, t, r)\, dt = -2\int_{x+\delta}^{\infty} [rP(x, t, r)]_r\, dt \to 0$ uniformly on compact subsets for any fixed $\delta > 0$, and similarly $\int_{-\infty}^{x-\delta} \Lambda P(x, t, r)\, dt \to 0$ uniformly in compact subsets, it is sufficient to restrict ourselves to $\int_{a_2}^{b_2} \Lambda P(x, t, r) H_2(t)\, dt$.

$$\begin{aligned} \int_{a_2}^{b_2} \Lambda P(x, t, r) H_2(t)\, dt &= \int_{a_2}^{b_2} [P_{tt}(x, t, r) - (t^2+1)P(x, t, r)] H_2(t)\, dt \\ \text{(40)} \qquad &= \int_{a_2}^{b_2} P_{tt}(x, t, r) H_2(t)\, dt - \int_{a_2}^{b_2} (t^2+1) P(x, t, r) H_2(t)\, dt \\ &= H_2(b_2) P_t(x, b_2, r) - H_2(a_2) P_t(x, a_2, r) - H_2'(b_2) P(x, b_2, r) \\ &\quad + H_2'(a_2) P(x, a_2, r) + \int_{a_2}^{b_2} P(x, t, r) \Lambda H_2(t)\, dt. \end{aligned}$$

The integral in (40) is zero on $[a_1, b_1]$ by (38). The first two terms in (40) go to zero uniformly on $[a_1, b_1]$ by (29), since b_2 is not contained in $[a_1, b_1]$, and $P_t(x, t, r) \to 0$ uniformly for $|x - t| \geqq \delta > 0$.

Thus (39) will be proved if we show

(41) $H_2'(b_2)P(x, b_2, r)$ and $H_2'(a_2)P(x, a_2, r) \to 0$ uniformly on $[a_1, b_1]$. Since $G(x, r) \to G(x)$ uniformly in x, and $G(x)$ is bounded, we have

$$\lim_{r\to 1} \int_0^x G(t, r)\, dt = \int_0^x \lim_{r\to 1} G(t, r)\, dt = \int_0^x G(t)\, dt.$$

Hence

$$\begin{aligned}\lim_{r\to 1} \int_0^x \sum_{n=0}^{\infty} \frac{a_n}{(2n+2)^2} \Phi_n'(t) r^n &= \lim_{r\to 1} \sum_{n=0}^{\infty} \int_0^x \frac{\Phi_n'(t) r^n}{(2n+2)^2}\, dt \\ &= \lim_{r\to 1} \left[\sum_{n=0}^{\infty} \frac{a_n}{(2n+2)^2} \Phi_n(x) r^n - \sum_{n=0}^{\infty} \frac{a_n}{(2n+2)^2} \Phi_n(0) r^n\right] \\ &= H(x) - H(0).\end{aligned}$$

This means $H(x)-H(0)=\int_0^x G(t)\, dt$ or $H'(x)=G(x)$. Defining

$$G_1(x, r) = \Omega\left[\frac{\partial}{\partial x} F_1(x, r)\right]$$

we get $G_1(x, r) \to G_1(x)$ uniformly, where $G_1(x)$ is bounded and continuous.

Setting $G_2(x)=G(x)-G_1(x)$ we see that $G_2(x)$ is a bounded and continuous function since $G(x)$ and $G_1(x)$ are. Thus (41) is proved if we can show

$G_2(b_2)P(x, b_2, r)$ and $G_2(a_2)P(x, a_2, r)$ converge uniformly to zero on $[a_1, b_1]$ as $r \to 1$.

But $G_2(x)$ is bounded and $b_2 \notin [a_1, b_1]$. Thus $P(x, b_2, r)$ and $P(x, a_2, r) \to 0$ uniformly on $[a_1, b_1]$. This proves (39). Since $\Lambda H_2(x, r)=F_2(x, r)$ we have:

$F(x, r)-F_1(x, r) \to 0$ uniformly on $[a_1, b_1]$, which completes the proof that $F(x, r) \to F(x)$ uniformly on compact subsets of (a, b).

Since $F_1(x) \in \mathscr{H}$, we have

$$F_1(x) \sim \sum_{n=0}^{\infty} \frac{a_n'}{2n+2} \Phi_n(x)$$

where $a_n'=\int_{-\infty}^{\infty} F_1(t)\Phi_n(t)\, dt$. Then

$$\begin{aligned}F_1(x, r) &= \int_{-\infty}^{\infty} F_1(t) P(x, t, r)\, dt = \int_{-\infty}^{\infty} F_1(t) \sum_{n=0}^{\infty} \Phi_n(x)\Phi_n(t) r^n\, dt \\ &= \sum_{n=0}^{\infty} \left[\int_{-\infty}^{\infty} \Phi_n(t) F_1(t)\, dt\right] \Phi_n(x) r^n \\ &= -\sum_{n=0}^{\infty} \frac{a_n'}{2n+2} \Phi_n(x) r^n,\end{aligned}$$

which is consistent with our previous notation.

By [2, Theorem 4] we have

$$\Lambda^* F_1(x) \geqq \liminf_{r\to 1} \sum_{n=0}^{\infty} a_n' \Phi_n(x) r^n = \liminf_{r\to 1} [-2rF_1(x, r)]_r. \tag{42}$$

Since $F_1(x)=F(x)$ on $[a_3, b_3]$, we have

$$\Lambda^* F_1(x) = \Lambda^* F(x) \quad \text{for } x \in [a_1, b_1]. \tag{43}$$

Repeating the identical argument which showed $\Lambda H_2(x, r) \to 0$ uniformly on $[a_1, b_1]$ as $r \to 1$, and using the properties that $\Lambda P(x, t, r)$ and $[\Lambda P(x, t, r)]_t$ converge uniformly to zero for $|x-t| \geqq \delta > 0$, we have $\Lambda[\Lambda H_2(x, r)] \to 0$ uniformly on $[a_1, b_1]$.

This means $[rF(x, r)]_r - [rF_1(x, r)]_r \to 0$ uniformly on compact subsets of (a, b). This gives, for $x_0 \in (a, b)$,

$$\liminf_{r\to 1} -2[rF(x_0, r)]_r = \liminf_{r\to 1} -2[rF_1(x_0, r)]_r. \tag{44}$$

But

$$\liminf_{r\to 1} -2[rF(x_0, r)]_r = f_*(x_0). \tag{45}$$

Thus combining (42), (43), (44), and (45) we have $\Lambda^* F(x) \geqq f_*(x)$. The proof that $f^*(x) \geqq \Lambda_* F(x)$ follows by a change of sign.

Λ-convex functions. The function $F(x)$ is said to be Λ-convex in (a, b) if the equations $y(c)=F(c)$ and $y(d)=F(d)$ for $a<c<d<b$ imply that $F(x) \leqq y(x)$ for $c<x<d$, where $y(x)$ is a solution of (17).

Generalized convex functions of this type have been studied by Beckenbach and Bing [8][9]. In particular, if F is Λ-convex in (a, b), then F is continuous in (a, b) and $F(a+)$, $F(b-)$ exist (as finite numbers or $\pm\infty$).

LEMMA 11. *Let $f(x, r)=\sum_{n=0}^{\infty} a_n \Phi_n(x) r^n$ exist for $-\infty<x<+\infty$ and $0 \leqq r<1$. Suppose that $|f(x, r)| \leqq \varepsilon(r)/(1-r)$ for $-\infty<x<+\infty$ and $0 \leqq r<1$, where $\varepsilon(r)$ is bounded and $\varepsilon(r)=o(1)$ as $r \to 1$, and that there is $y_1 \in \mathscr{H}$ such that $-\infty<y_1(x) \leqq f_*(x) \leqq f^*(x)<+\infty$ for all x. Set $F(x, r)=-\sum_{n=0}^{\infty}(a_n/(2n+2))\Phi_n(x) r^n$ for $0 \leqq r<1$, and suppose $F(x, r) \to F(x)$, where $F(x)$ is a continuous function on (a, b). Then $F(x)$ is continuous in $[a, b]$.*

Proof. It will be shown that:

(46) $F(a+)$ exists (finite or $\pm\infty$) and

(47) $F(a)=F(a+)$ and is finite.

By symmetry we then have $F(x)$ is continuous at b. By hypothesis we have $-\infty<y(x) \leqq f_*(x) \leqq f^*(x)<+\infty$. Let $u(x)$ be a function chosen as in [2, p. 396] such that

$u(x)$ is upper semicontinuous,

$u(x) \leqq y(x)<+\infty$ for all x, and

$u(x) \in \mathscr{H}$.

Set $W(x)=F(x)-\Omega u(x)$. Then $\Lambda^* W(x) \geqq \Lambda^* F(x)-\Lambda^* \Omega u(x) \geqq f_*(x)-u(x) \geqq y(x)-u(x) \geqq 0$ by [2, p. 392] and Lemma 10.

Applying [2, p. 395], we have $W(x)$ is Λ-convex in (a, b). Since W is Λ-convex to the right of a, this means

$$W(a+) \text{ exists as a finite number or } \pm\infty.$$

By the continuity of $\Omega u(x)$, this proves (46). To prove (47), we select $a<a'<b'<b$. Since

$$G(x, r) = \sum_{n=0}^{\infty} \frac{a_n}{(2n+2)^2} \Phi_n'(x) r^n \qquad \text{for } 0 \leqq r < 1,$$

and $\Phi_n(x)$ satisfies (16),

$$\begin{aligned} G_x(x, r) &= \sum_{n=0}^{\infty} \frac{a_n}{(2n+2)^2} \Phi_n''(x) r^n \\ &= -\sum_{n=0}^{\infty} \frac{a_n}{2n+2} \Phi_n(x) r^n + (x^2+1) \sum_{n=0}^{\infty} \frac{a_n}{(2n+2)^2} \Phi_n(x) r^n \\ &= F(x, r) + (x^2+1) H(x, r). \end{aligned}$$

Thus

$$G(a+h, r) = \int_{a'}^{a+h} F(t, r)\, dt + \int_{a'}^{a+h} (t^2+1) H(t, r)\, dt + G(a', r).$$

Taking the limit as $r \to 1$, and applying Lemmas 7 and 10 we have

$$G(a+h) - G(a') = \int_{a'}^{a+h} F(t)\, dt + \int_{a'}^{a+h} (t^2+1) H(t)\, dt.$$

Since $F(a+)$ exists (as finite or $\pm\infty$) we can take the limit as $a' \to a$ and get

$$G(a+h) - G(a) = \int_{a}^{a+h} F(t)\, dt + \int_{a}^{a+h} (t^2+1) H(t)\, dt.$$

Thus

$$\lim_{h \to 0^+} \frac{G(a+h) - G(a)}{h} = F(a+) + (a^2+1) H(a)$$

and this must be finite or $\pm\infty$.

But G is smooth by Lemma 7, which means that the right-hand and left-hand limits must agree. Therefore $G'(a) = F(a+) + (a^2+1)H(a)$. Applying Lemma 9

$$\begin{aligned} G'(a) &= \lim_{r \to 1} [F(a, r) + (a^2+1) H(a, r)] \\ &= F(a) + (a^2+1) H(a). \end{aligned}$$

This means $F(a) = F(a+)$. By Lemma 8, $F(a)$ is everywhere finite, proving (47).

LEMMA 12. *Let $f(x, r) = \sum_{n=0}^{\infty} a_n \Phi_n(x) r^n$ exist for $-\infty < x < +\infty$ and $0 \leqq r < 1$. Suppose that $|f(x, r)| \leqq \varepsilon(r)/(1-r)$ for $-\infty < x < +\infty$ and $0 \leqq r < 1$, where $\varepsilon(r)$ is* *bounded and* $\varepsilon(r)=o(1)$ *as* $r\to 1$, *and that there is* $y_1\in\mathscr{H}$ *such that* $-\infty<y_1(x)\leqq f_*(x)\leqq f^*(x)<+\infty$ *for all x. Set*

$$F(x,r) = -\sum_{n=0}^{\infty}\frac{a_n}{2n+2}\Phi_n(x)r^n \quad \textit{for } 0\leqq r<1.$$

Then $F(x)$ *is continuous in* $(-\infty,\infty)$.

Proof. Let $R>0$. It is sufficient to show $F(x)$ is continuous on $(-R,R)$. In Lemma 8 we saw that the function

$$F(x,r) = -\sum_{n=0}^{\infty}\frac{a_n}{2n+2}\Phi_n(x)r^n, \qquad 0\leqq r<1,$$

was Poisson summable at all points x to a function $F(x)$. Let E be the set of points at which $F(x)$ is discontinuous. We propose to show that E is the empty set.

Choose increasing sequence $\{r_n\}_{n=1}^{\infty}$ with $r_1=0$ and $r_n\to 1$ as $n\to\infty$, and with the property that if $r_n\leqq r\leqq r_{n+1}$, then $|f(x,r)-f(x,r_n)|\leqq 1$ for $|x|\leqq 2R$. Since $\limsup_{n\to\infty}|f(x,r_n)|\leqq f^*(x)<+\infty$, given any closed nondegenerate interval $\bar{J}$ contained in the interior of $(-R,R)$ there exists, by [10, Lemma 4, p. 645], a constant M and a closed nondegenerate subinterval $\bar{J}_1$ of $\bar{J}$ such that $|f(x,r_n)|\leqq M$ for x in $\bar{J}_1$. But then $|f(x,r)|\leqq M+1$ for $0\leqq r<1$ and x in $\bar{J}_1$. This fact implies that $F(x,r)\to F(x)$ uniformly for x in $\bar{J}_1$, and therefore that $F(x)$ is continuous in $\bar{J}_1$. We conclude that the set E is nondense in $(-R,R)$.

Suppose E contains an isolated point z_0. Then there is an $h>0$ such that $-R<z_0-h<z_0+h<R$ and such that $F(x)$ is continuous in each of the open intervals (z_0-h,z_0) and (z_0,z_0+h). Applying Lemma 11, $F(x)$ is continuous at z_0, thus E contains no isolated points.

If E is not vacuous, its closure $\bar{E}$ is perfect. We again apply [10, Lemma 4, p. 645]. Let $\pi=J\cdot\bar{E}$, where J is a segment, be a portion of $\bar{E}$ on which $F(x)$ is continuous. Let (a,b) be a segment contiguous to π. On (a,b), $F(x)$ is continuous. Thus by Lemma 11, $F(x)$ is continuous on $[a,b]$. If $x_0\in E$ is a left-hand endpoint or right-hand endpoint of one of the contiguous intervals of π, this gives $F(x)$ is continuous at x_0. Thus we may suppose x_0 is the limit of left-hand endpoints of contiguous intervals. Applying Lemma 11 to each contiguous interval (c,d), we have $F(c)=F(c+)$ and $F(d)=F(d-)$. Defining $W(x)=F(x)-\Omega u(x)$ as in Lemma 11, we have $W(x)$ is Λ-convex in each interval and $W(c+)=W(c)$, $W(d-)=W(d)$. It follows that if x_0 is any point of π, then $W(x)$ is upper semicontinuous at x_0. Then $W(x)$ is upper semicontinuous in J, which implies that $W(x)$ is Λ-convex and therefore continuous in J. Thus E is vacuous and $F(x)$ is continuous in $(-R,R)$.

The above proof is very similar to the analogous theorem for trigonometric series [3, p. 355].

CHAPTER II

Major theorems.

THEOREM I. *Let the series* $\sum_{n=0}^{\infty} a_n \Phi_n(x) r^n$ *converge, for* $0 \leqq r < 1$, *to* $f(x, r)$. *Suppose that*

(i) $|f(x, r)| = o(1/(1-r))$ *uniformly in x as* $r \to 1$;

(ii) *there is a function* $y_1 \in \mathscr{H}$ *such that* $-\infty < y_1(x) \leqq f_*(x) \leqq f^*(x) < +\infty$ *for all x*;

(iii) *there is a function* $y_2 \in \mathscr{H}$ *such that* $-\infty < y_2(x) \leqq F(x)$ *for all x.*

Then the series $\sum_{n=0}^{\infty} a_n \Phi_n(x)$ *is Poisson summable almost everywhere, and is the Hermite series of its Poisson sum.*

Proof. As in Lemma 10, we set

$$H(x, r) = \Omega F(x, r) = \sum_{n=0}^{\infty} \frac{a_n}{(2n+2)^2} \Phi_n(x) r^n.$$

In (34) we showed $H(x, r)$ converged to the continuous and bounded function $H(x)$ as $r \to 1$. Repeating the argument of (37), we have

$$H(x) \sim \sum_{n=0}^{\infty} \frac{a_n}{(2n+2)^2} \Phi_n(x).$$

By [2, Theorem 5] and hypothesis (iii), the series $-\sum_{n=0}^{\infty} (a_n/(2n+2))\Phi_n(x)$ is Poisson summable almost everywhere (we already proved in Lemma 8 this was everywhere to $F(x)$) and

$$F(x) \sim \sum_{n=0}^{\infty} \frac{a_n}{2n+2} \Phi_n(x). \tag{48}$$

Again applying [2, Theorem 5] to (48), because of Lemma 12 and hypothesis (ii), we have that the series $\sum_{n=0}^{\infty} a_n \Phi_n(x)$ is Poisson summable almost everywhere and is the Hermite series of its Poisson sum.

This result extends Rudin's results [2, Theorem 6], because the condition $a_n = o(n^{1/4})$ implies $|f(x, r)| = o(1/(1-r))$ uniformly in x as $r \to 1$. In fact, since

$$P(x, x, r) = \sum_{n=0}^{\infty} \Phi_n(x)\Phi_n(x) r^n = O(1/(1-r)^{1/2})$$

uniformly in x as $r \to 1$, and $\sum_{n=0}^{\infty} o(n^{1/2}) r^n = o((1/(1-r))^{3/2})$ as $r \to 1$, we have

$$\begin{aligned} |f(x, r)| &= \left| \sum_{n=0}^{\infty} (a_n r^{n/2})(\Phi_n(x) r^{n/2}) \right| \leqq \left(\sum_{n=0}^{\infty} a_n^2 r^n \right)^{1/2} \left(\sum_{n=0}^{\infty} \Phi_n^2(x) r^n \right)^{1/2} \\ &\leqq \left[o\left(\frac{1}{1-r}\right)^{3/2} \right]^{1/2} \left[O\left(\frac{1}{(1-r)^{1/2}}\right) \right]^{1/2} = o\left(\frac{1}{1-r}\right) \end{aligned}$$

uniformly in x as $r \to 1$. If $a_n = o(n^{1/4})$, the condition that $F(x) \in \mathscr{H}$ follows from the Riesz-Fischer Theorem.

THEOREM II. *Let the series* $\sum_{n=0}^{\infty} a_n \Phi_n(x) r^n$ *converge, for* $0 \leqq r < 1$, *to* $f(x, r)$. *Suppose that*

(i) $|f(x, r)| = o(1/(1-r))$ *uniformly in* x *as* $r \to 1$;

(ii) $\lim_{r \to 1} f(x, r) = 0$ *for all* x.

Then $a_n = 0$ *for all* n.

Proof. By Lemmas 10 and 12, $F(x, r) = -\sum_{n=0}^{\infty} (a_n/(2n+2)) \Phi_n(x) r^n$ converges as $r \to 1$ uniformly on compact subsets. By Lemma 10(b) we also have that $\Lambda^* F(x) \geqq 0 \geqq \Lambda_* F(x)$. Applying [2, Corollary 6.3], $F(x)$ is a solution to (17) for all x, that is $F''(x) - (x^2+1)F(x) = 0$. By (16),

$$\Phi_n(x) = (x^2+1)\frac{\Phi_n(x)}{2n+2} - \frac{\Phi_n''(x)}{2n+2}.$$

Thus

$$-(x^2+1)\sum_{n=0}^{\infty} \frac{a_n}{(2n+2)^2} \Phi_n(x) r^n + \sum_{n=0}^{\infty} \frac{a_n}{(2n+2)^2} \Phi_n''(x) r^n \to F(x)$$

uniformly on compact subsets. Integrating twice,

$$\sum_{n=0}^{\infty} \frac{a_n r^n}{(2n+2)^2} \int_0^x \int_0^s \Phi_n''(t)\, dt\, ds - \int_0^x \int_0^s (t^2+1) H(t, r)\, dt\, ds$$

converges to $\int_0^x \int_0^s F(t)\, dt\, ds$, where $H(x, r) = \Omega F(x, r)$, $(0 \leqq r < 1)$, as defined in Lemma 10. Since

$$\int_0^x \int_0^s \Phi_n''(t)\, dt\, ds = \int_0^x [\Phi_n'(s) - \Phi_n'(0)]\, ds = \Phi_n(x) - \Phi_n(0) - x\Phi_n'(0),$$

$$\begin{aligned}(49)\quad & \sum_{n=0}^{\infty} \frac{a_n}{(2n+2)^2} \Phi_n(x) r^n - \sum_{n=0}^{\infty} \frac{a_n}{(2n+2)^2} \Phi_n(0) r^n - x \sum_{n=0}^{\infty} \frac{a_n}{(2n+2)^2} \Phi_n'(0) r^n \\ & - \int_0^x \int_0^s (t^2+1) H(t, r)\, dt\, ds \quad \text{converges to} \int_0^x \int_0^s F(t)\, dt\, ds.\end{aligned}$$

But (49) is known to converge to

$$H(x) - H(0) - xG(0) - \int_0^x \int_0^s (t^2+1) H(t)\, dt\, ds.$$

Therefore

$$(50)\qquad H(x) - H(0) - xG(0) - \int_0^x \int_0^s (t^2+1) H(t)\, dt\, ds = \int_0^x \int_0^s F(t)\, dt\, ds.$$

Since H and G are bounded (Lemmas 7 and 10), the left-hand side of (50) is $O(x^4)$ for x large.

Since the function $\beta(x) = e^{x^2/2} \int_{-\infty}^{x} e^{-u^2}\, du$ and $\beta(-x) = e^{x^2/2} \int_{-\infty}^{-x} e^{-u^2}\, du$ are linearly independent solutions to (17), we have $F(x) = c_1\beta(x) + c_2\beta(-x)$. Let

$$g_1(x) = \frac{c_1}{x^4} \int_0^x \int_0^s \beta(t)\, dt\, ds \quad \text{and} \quad g_2(x) = \frac{c_2}{x^4} \int_0^x \int_0^s \beta(-t)\, dt\, ds.$$

As $x \to +\infty$, $g_1(x) \to +\infty$ and $g_2(x) \to 0$. So if $\int_0^x \int_0^s F(t)\,dt\,ds$ is $O(x^4)$, then $c_1=0$. As $x \to -\infty$, $g_2(x) \to +\infty$ and $g_1(x) \to 0$. So if $\int_0^x \int_0^s F(t)\,dt\,ds$ is $O(x^4)$, then $c_2=0$. Thus $F(x)=0$ for all x. Theorem I now applies and gives us $0 \sim \sum_{n=0}^{\infty} a_n \Phi_n(x)$ or $a_n=0$ for all n.

We cannot replace the condition $o(1/(1-r))$ by $O(1/(1-r))$ in Theorems I and II. To illustrate this, if we differentiate $e^{x^2/2}P(x, 0, r)$ we get

$$f(x, r) = \sum_{n=0}^{\infty} (2n+2)^{1/2}\Phi_{n+1}(0)\Phi_n(x)r^n = \frac{-2rx}{\pi^{1/2}(1-r^2)^{3/2}} \exp\left\{-\frac{x^2}{2}\left(\frac{1-r^2}{1+r^2}\right)\right\} \qquad (0 \leqq r < 1).$$

This is a series of Hermite functions for which $f(x, r) \to 0$ for all x, $F(x) \in \mathscr{H}$, and $f(x, r)=O(1/(1-r))$ uniformly in x as $r \to 1$. This series is clearly not the zero series, and we conclude the above condition is in a certain sense best possible.

Bibliography

1. J. Indritz, *Methods in analysis*, Macmillan, New York, 1963.

2. W. Rudin, *Uniqueness theory for Hermite series*, Trans. Amer. Math. Soc. **70** (1951), 387–403.

3. A. Zygmund, *Trigonometric series*, Vol. I. Cambridge Univ. Press, New York, 1959.

4. V. Shapiro, *The uniqueness of solutions of the heat equation in an infinite strip*, Trans. Amer. Math. Soc. **125** (1966), 326–361.

5. G. Sansone, *Orthogonal functions*, Interscience, New York, 1959.

6. G. N. Watson, *Notes on generating functions of polynomials.* II: *Hermite polynomials*, J. London Math. Soc. **8** (1933), 194–199.

7. G. Szegö, *Orthogonal polynomials*, Amer. Math. Soc. Colloq. Publ., Vol. 23, Amer. Math. Soc., Providence, R. I., 1939; rev. ed. 1967.

8. E. F. Beckenbach, *Generalized convex functions*, Bull. Amer. Math. Soc. **43** (1937), 363–371.

9. E. F. Beckenbach and R. H. Bing, *On generalized convex functions*, Trans. Amer. Math. Soc. **58** (1945), 220–230.

10. V. Shapiro, *The uniqueness of functions harmonic in the interior of the unit disc*, Proc. London Math. Soc. **13** (1963), 639–652.

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